Chapter 9

Fractional Quantum Hall Effect

9.1 The Hall effect

The Hall effect, discovered in 1879, occurs when a current carrying conducting plate, placed in the x-y plane, is subjected to a magnetic field B_z perpendicular to the plane. The magnetic field causes the electrons, of current density j_x moving at a velocity v_x, to experience a Lorentz force $v_x B_z$ which results in a voltage drop across the sample (see Fig. 9.1). The Hall conductance, which is a measure of the conductance across the sample is

$$\sigma_{xy} = \frac{j_x}{E_y} = \frac{nev_x}{v_x B} = \frac{ne}{B}, \tag{9.1}$$

where n is the number density. Therefore, classically, we would expect the Hall conductance to vary linearly with ne/B.

9.2 The integer quantum Hall effect

In the early 1970's, due to progress in the material sciences, new semiconductors were being produced which could be layered to form very flat, thin interfaces. By cooling these materials and applying an electric field perpendicular to the interface surface the electrons would be forced to situate themselves deep within quantum wells. The result was to quantize the motion of the electrons perpendicular to the interface. This severely limited the motion of the electrons constraining them to move, essentially, within only two-dimensions.

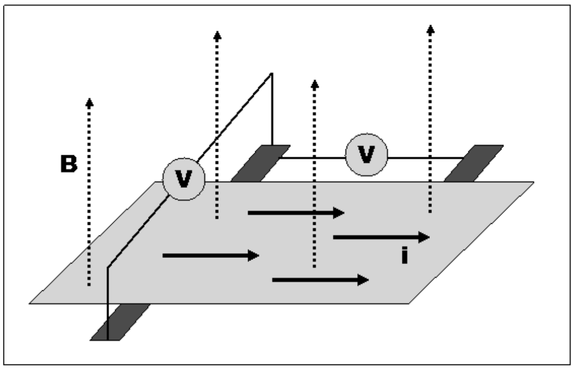

Figure 9.1: The Hall effect occurs when an electrical current is subjected to a transverse magnetic field. The Hall voltage (perpendicular to the current) drops due to the Lorentz force felt by the electrons.

In 1980, Klaus von Klitzing [103] found that at temperatures of only a few Kelvin and high magnetic field (3-10 Tesla), the Hall resistance did not vary linearly with the field. Instead, he found that it varied in a stepwise fashion (see Fig. 9.2). It was also found that where the Hall resistance was flat, the longitudinal resistance disappeared. This dissipation free transport looked very similar to superconductivity. The field at which the plateaus appeared, or where the longitudinal resistance vanished, quite surprisingly, was independent of the material, temperature, or other variables of the experiment, but only depended on a combination of fundamental constants -$\hbar/e^2$. The quantization of resistivity seen in these early experiments came as a grand surprise and would lead to a new international standard of resistivity, the Klitzing, defined as the Hall resistance measured at the fourth step.

The Integer Quantum Hall Effect (IQHE), as it later came to be known, may simply be explained in the context of non-interacting quantum mechanics. The Hamiltonian for a particle subjected to a magnetic field perpendicular to its direction of motion can be written down as:

$$H = \sum_{i}^{N} \frac{(\mathbf{p}_i - e\mathbf{A}(\mathbf{x}_i))^2}{2m}, \tag{9.2}$$

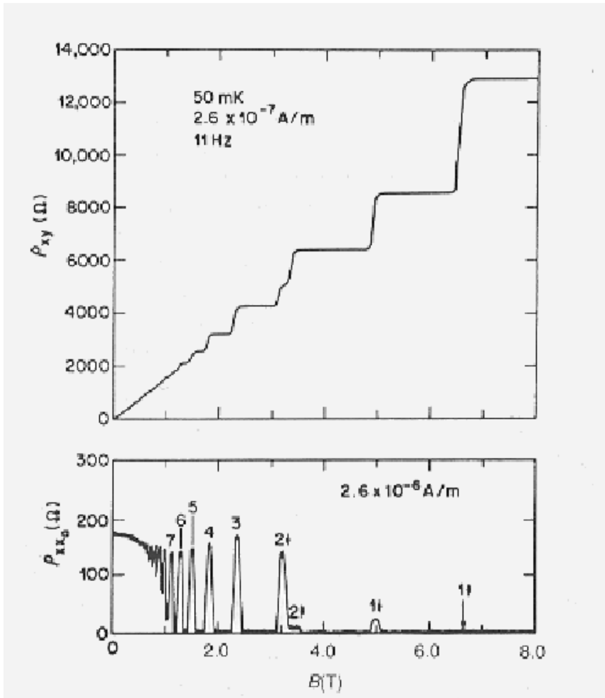

Figure 9.2: The top figure shows the stepwise behavior of the transverse resistivity and the bottom the longitudinal resistance, both as a function of magnetic field. Note, the plateaus(top) coincide with the dissipationless behavior(bottom) at each value of ν. (reprinted from reference [104])

with the vector potential $\mathbf{A}(\mathbf{x})$ chosen in the following gauge

$$\mathbf{A}(\mathbf{x}_i) = \frac{1}{2}B(y_i, -x_i), \qquad B = B\hat{z}. \tag{9.3}$$

The above Hamiltonian results in energy eigenvalues $E_{n,k_y} = (n+1/2)\hbar\omega$ for each "Landau level" (LL) which depend on the cyclotron frequency $\omega = eB/mc$ (see Appendix F). Since the LLs do not depend on the quantum number k_y, they are highly degenerate. This degeneracy, defined as the number of states per unit area, may be quantified by the relation $\rho_B = eB/hc$. This relation leads to a useful quantity for describing the IQHE, known as the filling factor:

$$\nu = \rho/\rho_B, \tag{9.4}$$

which is the number of electrons per Landau level and acts as a measure of the applied magnetic field.

In Fig. 9.2, the plateaus occur at each integer value of the filling factor ν. What occurs is that, as each of the degenerate states of a LL is filled, the conductivity decreases, or the resistivity increases, because fewer and fewer states remain unoccupied within that energy level. Once the LL level is completely full, a gap exists requiring a finite jump in energy to reach the next set of degenerate energy states (i.e., the next LL). However, due to impurities in a sample, localized states exist which may be filled, but do not contribute to the conductivity. This mechanism explains the stepwise behavior of the resistivity.

9.3 Fractional quantum Hall effect

By 1982, semiconductor technology had greatly advanced and it became possible to produce interfaces of much higher quality than where available only a few years before. That same year, Horst Stormer and Dan Tsui [105] repeated Klitzing's earlier experiments with much cleaner samples and higher magnetic fields. What they found was the same stepwise behavior as seen previously, but to everyone's surprise, steps also appeared at fractional filling factors $\nu = 1/3, 1/5, 2/5\ldots$ (see Fig. 9.3).

Since these fractional values occurred in the middle of the various highly degenerate Landau levels, where no gap was apparent, these observations could not be explained by the non-interacting quantum mechanical theory.

It would quickly be realized that these observations were a result of the many-body effects of interacting electrons. We should once again consider the Hamiltonian for a population of electrons in the presence of a transverse magnetic field, but now include the effects of interactions:

$$H = \sum_{i}^{N} \frac{(\mathbf{p}_i - e\mathbf{A}(\mathbf{x}_i))^2}{2m} + \sum_{i<j}^{N} \frac{e^2}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{9.5}$$

For the IQHE the potential energy $e^2/\bar{r}$, where $\bar{r}$ is the average electron spacing, was small compared to the cyclotron energy $\omega = eB/mc$ and could be neglected. Now, however, the kinetic term may be neglected so that $e^2/\bar{r}$ is no longer a small term and we are left with a problem which is intrinsically one of strong correlations.

9.4 The Laughlin variational wavefunction

Strongly correlated systems are notoriously difficult to understand, but in 1983, Robert Laughlin [106] proposed his now celebrated ansatz for a variational wavefunction which contained no free parameters:

$$\psi_L = \prod_{i<j}(z_i - z_j)^{2p+1} e^{-\sum_i \frac{|z_i|^2}{4l^2}}, \tag{9.6}$$

where the position of the i^{th} particle is given by the complex coordinate $z_i = x_i + iy_i$ and $l^2 = hc/eB$ is the magnetic length . The Laughlin wavefunction, as it would come to be known, gave an accurate description of all filling fractions $\nu = 1/(2p+1)$ and was shown to almost exactly match numerical ground state wavefunctions found for small FQHE systems. There are several key points that should be mentioned about this choice of wavefunction:

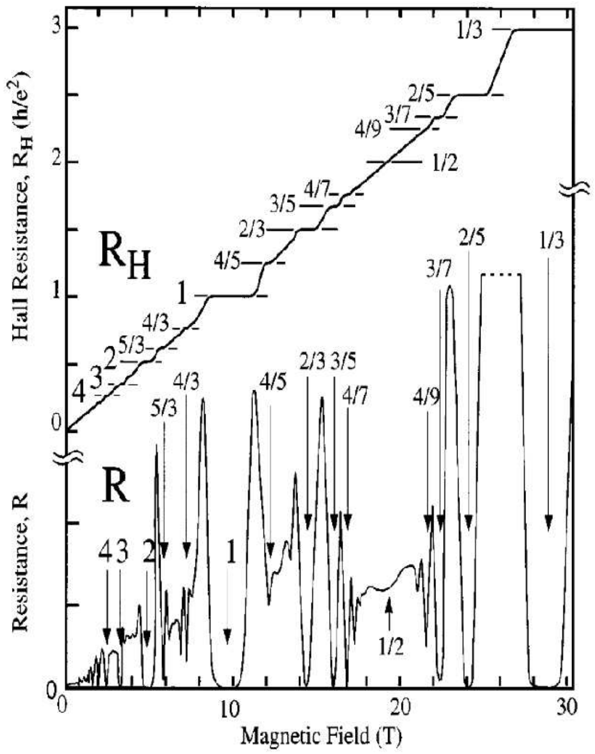

Figure 9.3: The figure shows the stepwise behavior of the transverse resistivity, superimposed with the longitudinal resistance, as a function of magnetic field. The same behavior as in figure 9.2 is seen except now at fractions of ν. (reprinted from reference [104])

- The wavefunction is extremely efficient at minimizing the Coulomb energy. Since the wavefunction has multiple zeros, occurring whenever two particles approach, the wavefunction results in a very uniform distribution. This distribution is much more efficient at lowering the Coulomb energy than, for example, a random distribution.

- The Laughlin wavefunction is the lowest energy state and is no longer degenerate, as one might naively expect. The state, therefore, is incompressible and will generate a finite energy gap for all excitations.

- The quasi-particles which result above the ground state carry with them a fractional charge.

One of the main reasons the Laughlin wavefunction is so celebrated is because it is a true, highly-correlated wavefunction. In contrast, the majority of many-body wavefunctions are simply Slater-determinants of single-particle wavefunctions. For example, the wavefunction for one filled Landau level can be written as proportional to:

$$\psi(z_1, z_2, \ldots z_N) \sim \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{N-1} & z_2^{N-2} & \cdots & z_N^{N-1} \end{vmatrix}, \tag{9.7}$$

where we have not bothered to write out the normalization or the usual decaying exponential factors. The Laughlin wavefunction, however, cannot be written in terms of a Slater determinant and is, therefore, something much more complicated than is often encountered. In order to explain the remaining filling fractions, such as $2/5, 3/7$... it was necessary to build off the work of Laughlin to create what has come to be known as the hierarchy picture.

9.5 Composite fermions and Chern-Simons theory

In 1989, Jainendra Jain [107] identified a suitable set of quasi-particles for the FQHE system, calling them "composite fermions". In terms of these quasi-particles, the FQHE behaves exactly as does the IQHE. When each of the composite fermion Landau levels is filled there exists an energy gap separating it from the next composite Landau level. This scheme was able to explain both the Laughlin fractions $1/3, 1/5\ldots$ as well as many of the remaining fractions $2/5, 3/7\ldots$. The idea of composite particles may be clarified by the illustrations of Figs. 9.4, 9.5, and 9.6.

A system of composite fermions may seem strange at first thought since the quasi-particles are composed of both particles (the electrons) and fields (the flux quanta). A field theory of this sort is known as a gauge field theory since it will require a gauge transformation to move from particles to quasi-particles. In the context of the FQHE, this transformation is known as the Chern-Simons theory [108]. We will discuss the Chern-Simons theory more in the next chapter where we apply it to describe a resonant gas of Bosons.

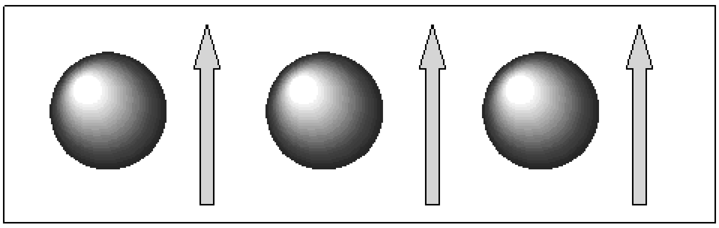

Figure 9.4: The IQHE may be depicted by the above illustration. For each particle (balls) there exists one associated flux quanta (arrows) and the particles are weakly interacting. This results in a filling fraction of $\nu = 1$.

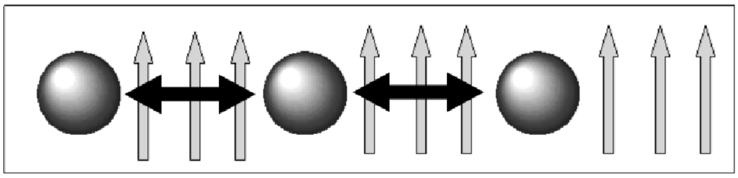

Figure 9.5: At a filling fraction of $\nu = 1/3$, for example, there exist three flux quanta for each particle, but the particles are now strongly interacting.

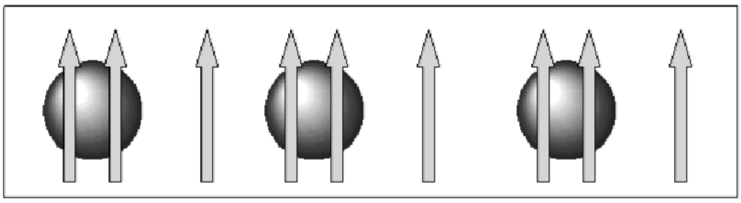

Figure 9.6: By associating two flux quanta with each particle, the system at $\nu = 1/3$ reduces to a weakly interacting system of composite particles (one particle and two flux quanta) at $\nu = 1$. We are left with the IQHE of figure 9.4, but in terms of composite fermions.