Topic 4: The Finite Potential Well

Outline:

- The quantum well
- The finite potential well (FPW)
- Even parity solutions of the TISE in the FPW
- Odd parity solutions of the TISE in the FPW
- Tunnelling into classically forbidden regions
- Comparison with the IPW

The AlGaAs-GaAs Quantum Well

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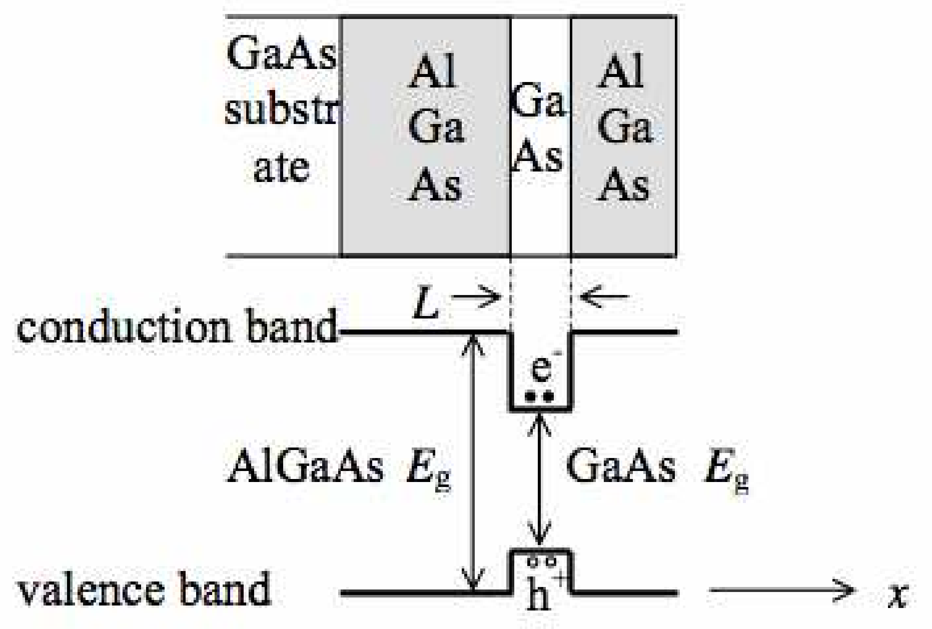
Example of a potential well:

Sandwich of GaAs and AlGaAs layers



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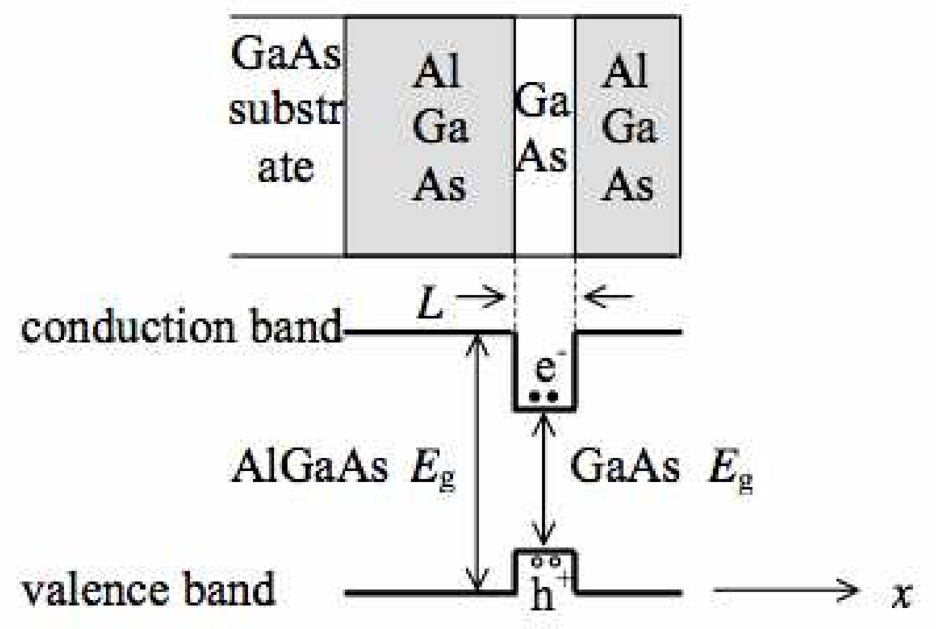

Constrained motion along the x-axis; free motion in the $y - z$ plane.

- $V_0 \to \infty$: 1-dimensional infinite potential well
- $V_0 < \infty$: 1-dimensional finite potential well

The Finite Potential Well

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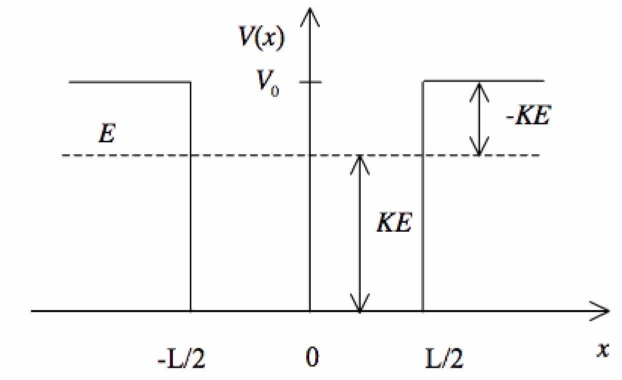
Consider a particle in the potential

$$V(x) = \begin{cases} V_0 & x < -\frac{L}{2} \\ 0 & -\frac{L}{2} \le x \le \frac{L}{2} \\ V_0 & x > \frac{L}{2} \end{cases}$$

$V_0 > 0$



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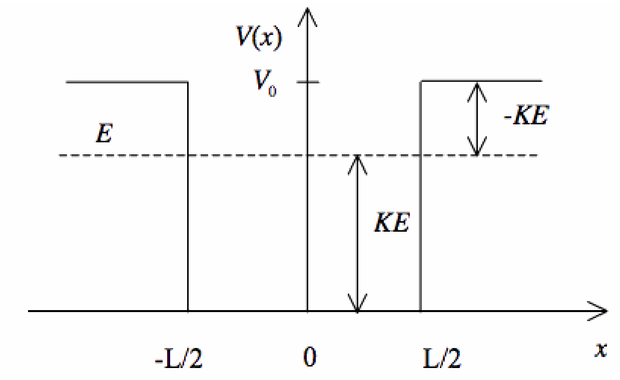

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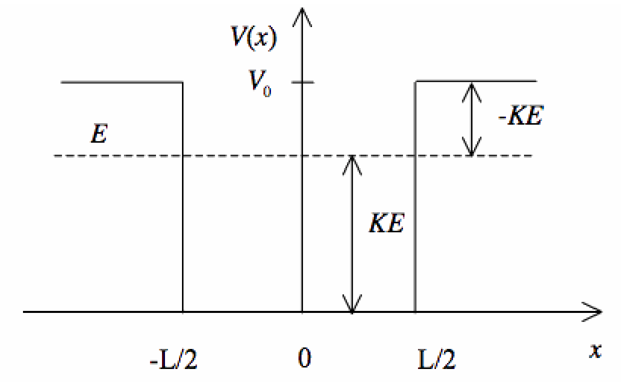

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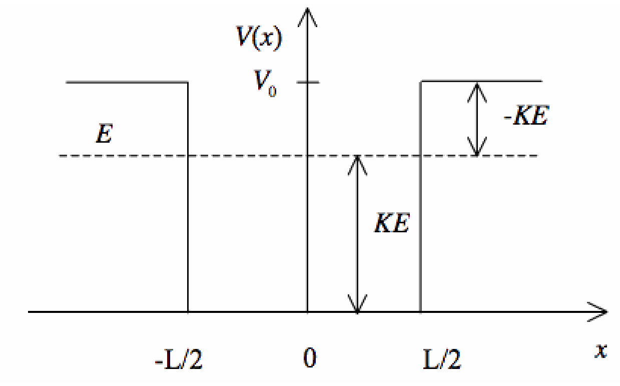

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Solve the TISE:

- $-\frac{L}{2} \leq x \leq \frac{L}{2}$ (Region I):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) \;\Rightarrow\; \psi''(x) = -k^2\psi(x) \qquad \boxed{k^2 = \frac{2mE}{\hbar^2} > 0}$$

solutions: $\psi_I(x) = A\sin kx + B\cos kx,$ (as for the IPW)

A, B – arbitrary constants

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- $x < -\frac{L}{2}$ (Region III):

  solutions $\psi_{III}(x) = Fe^{-\alpha x} + Ge^{\alpha x}$ (like in region II)

  F, G – arbitrary constants

  $\Rightarrow$ put $F = 0$, otherwise $\psi(x)$ not square integrable (blows up at large -ve x)

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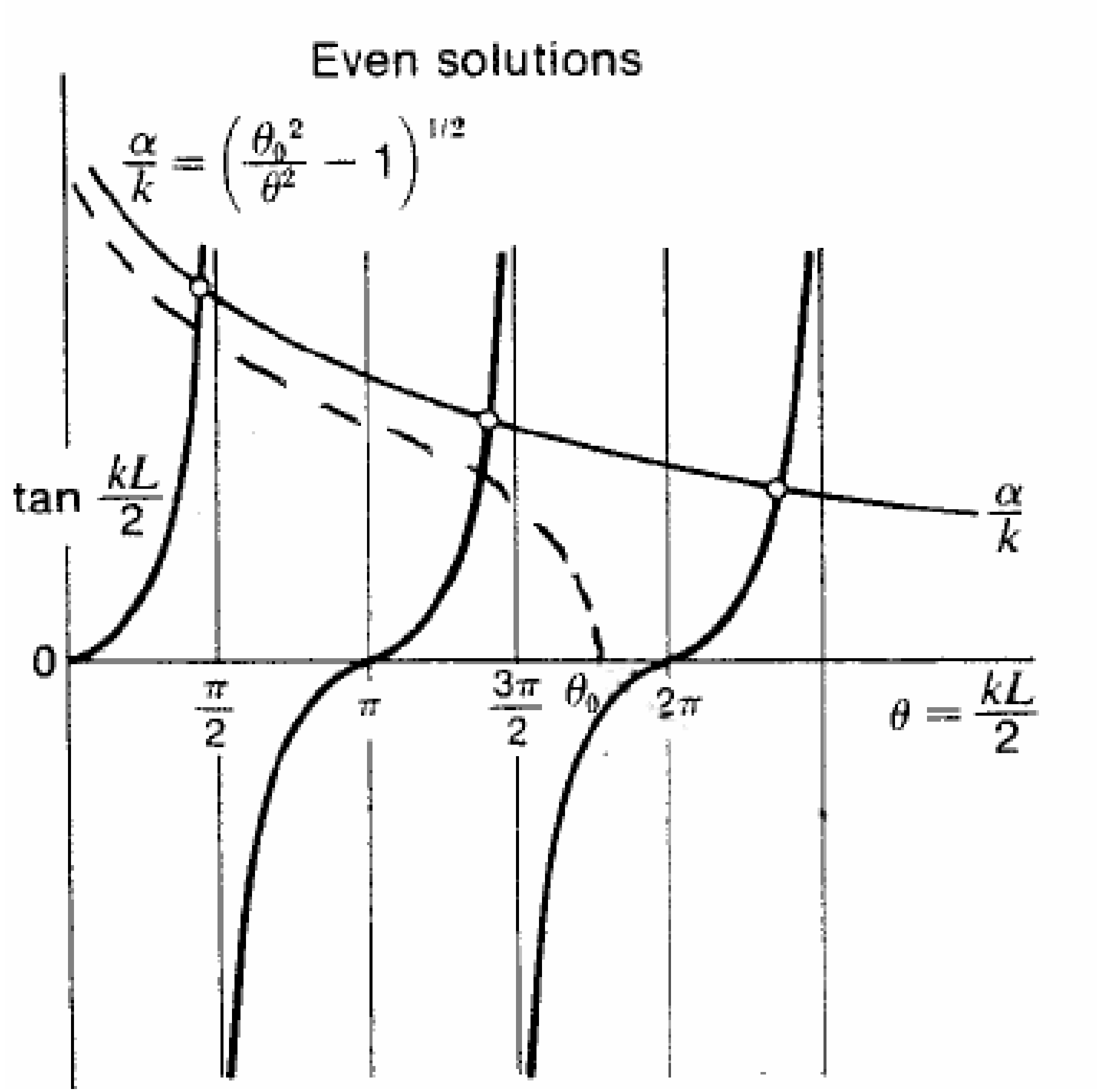

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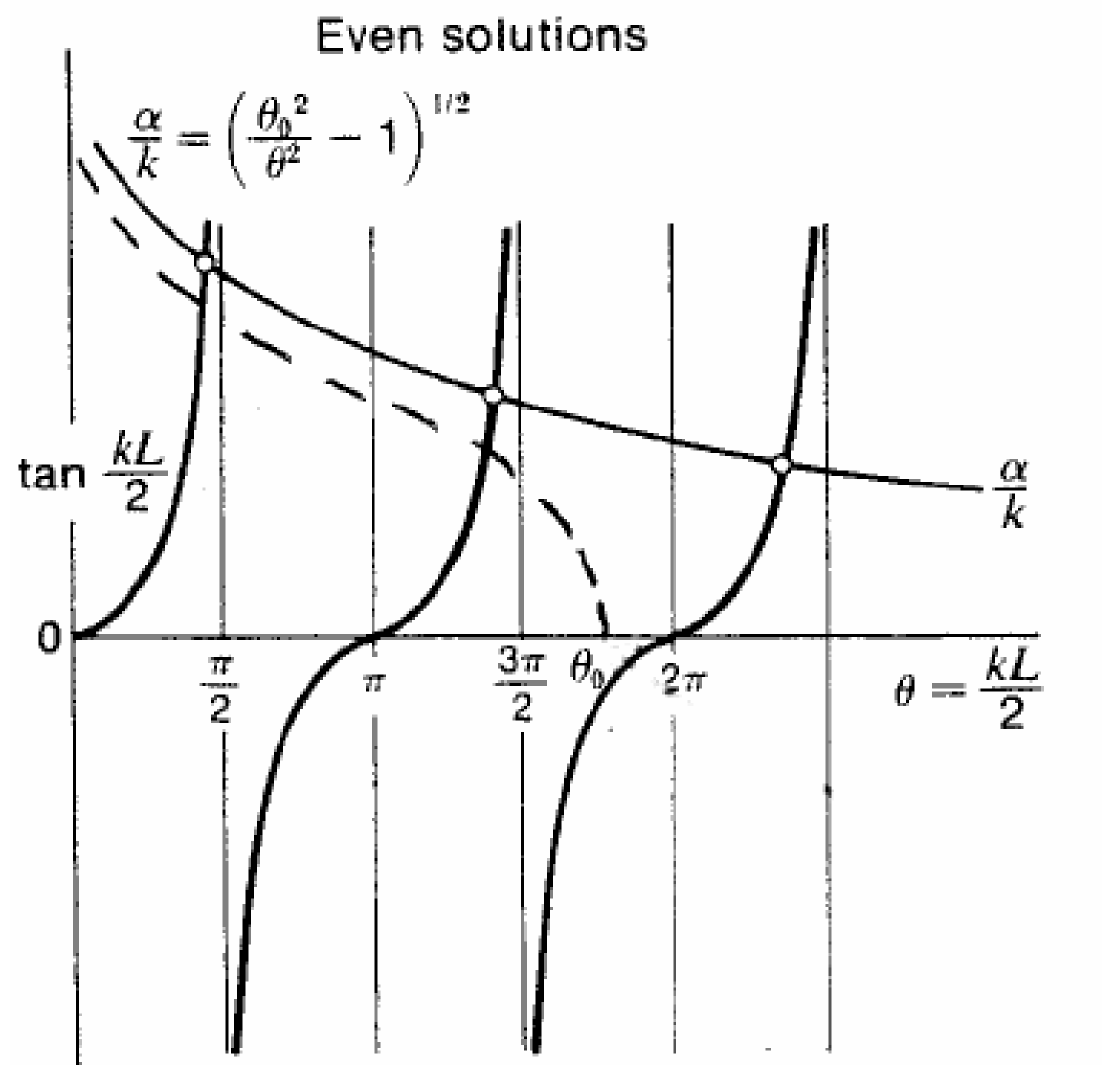

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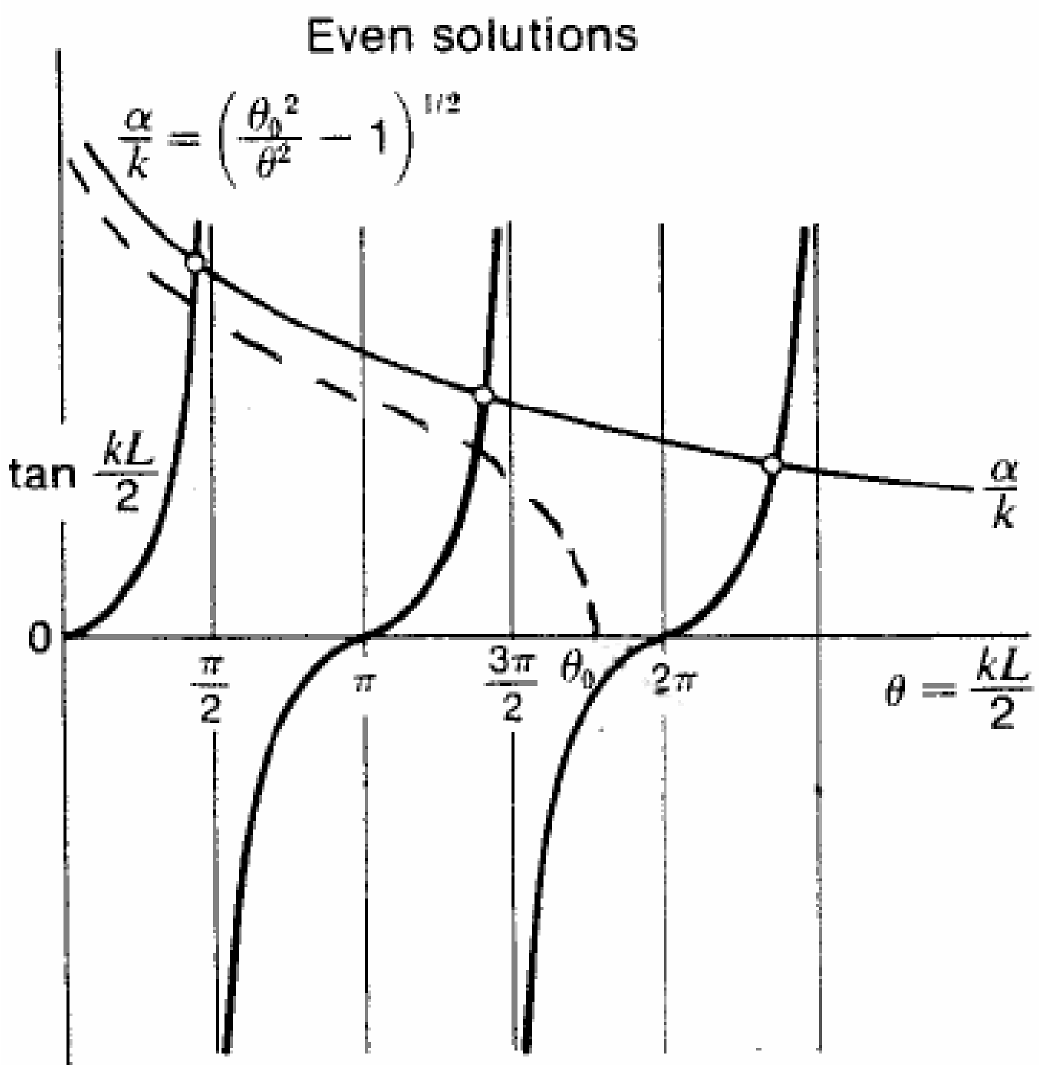

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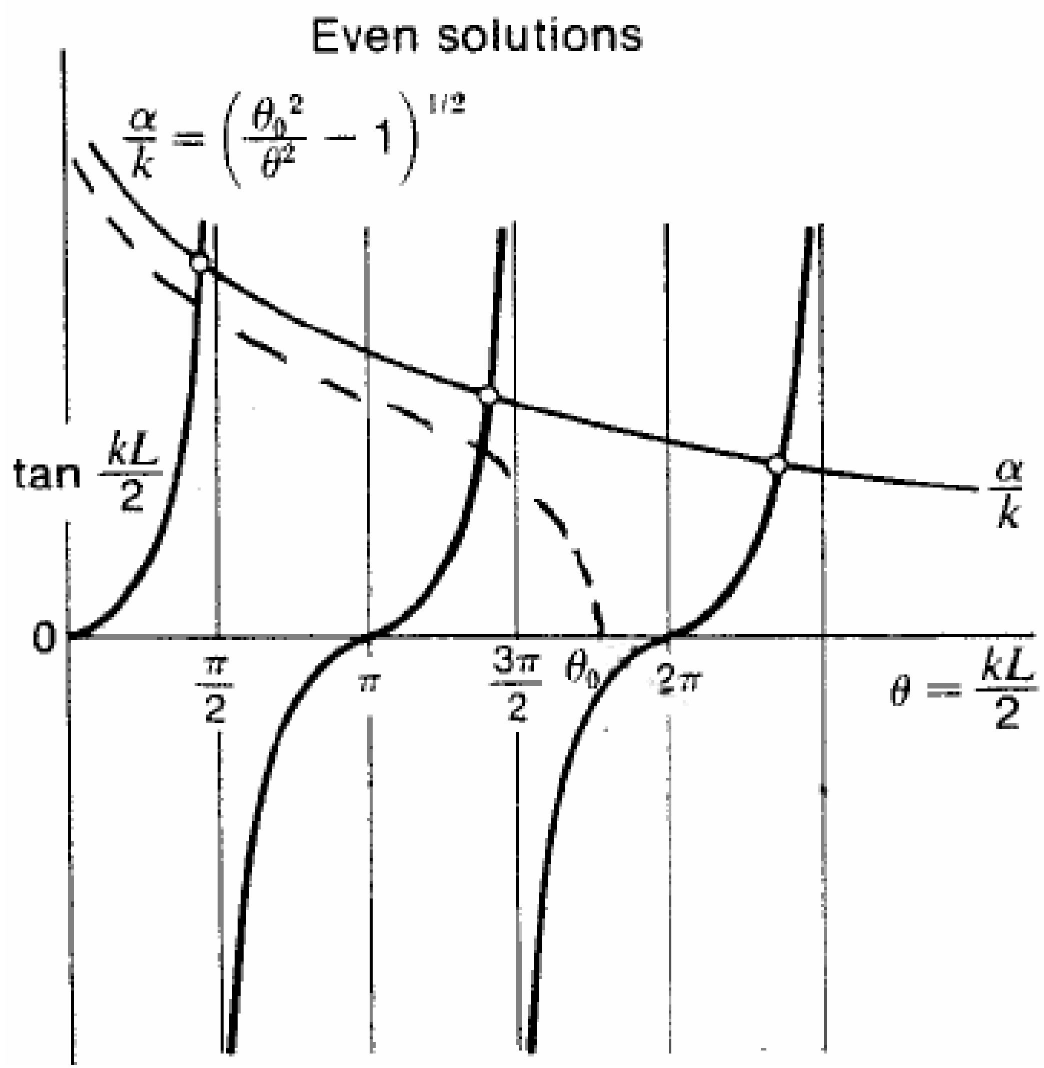

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$\Rightarrow$ bound states with discrete (quantized) energy

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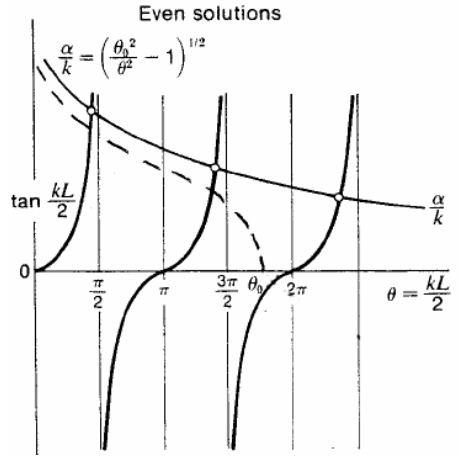

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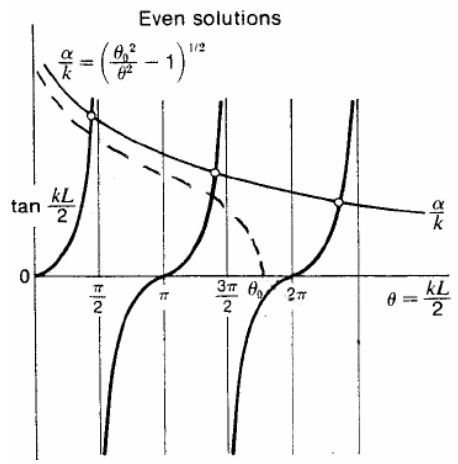

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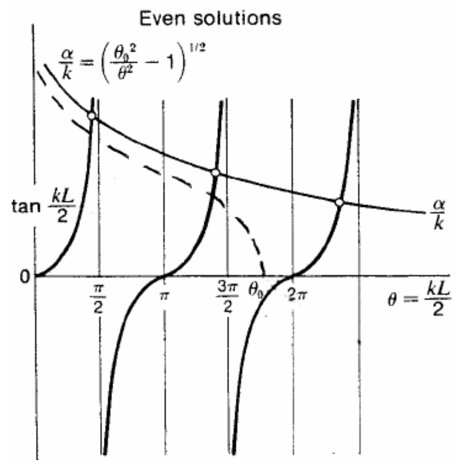

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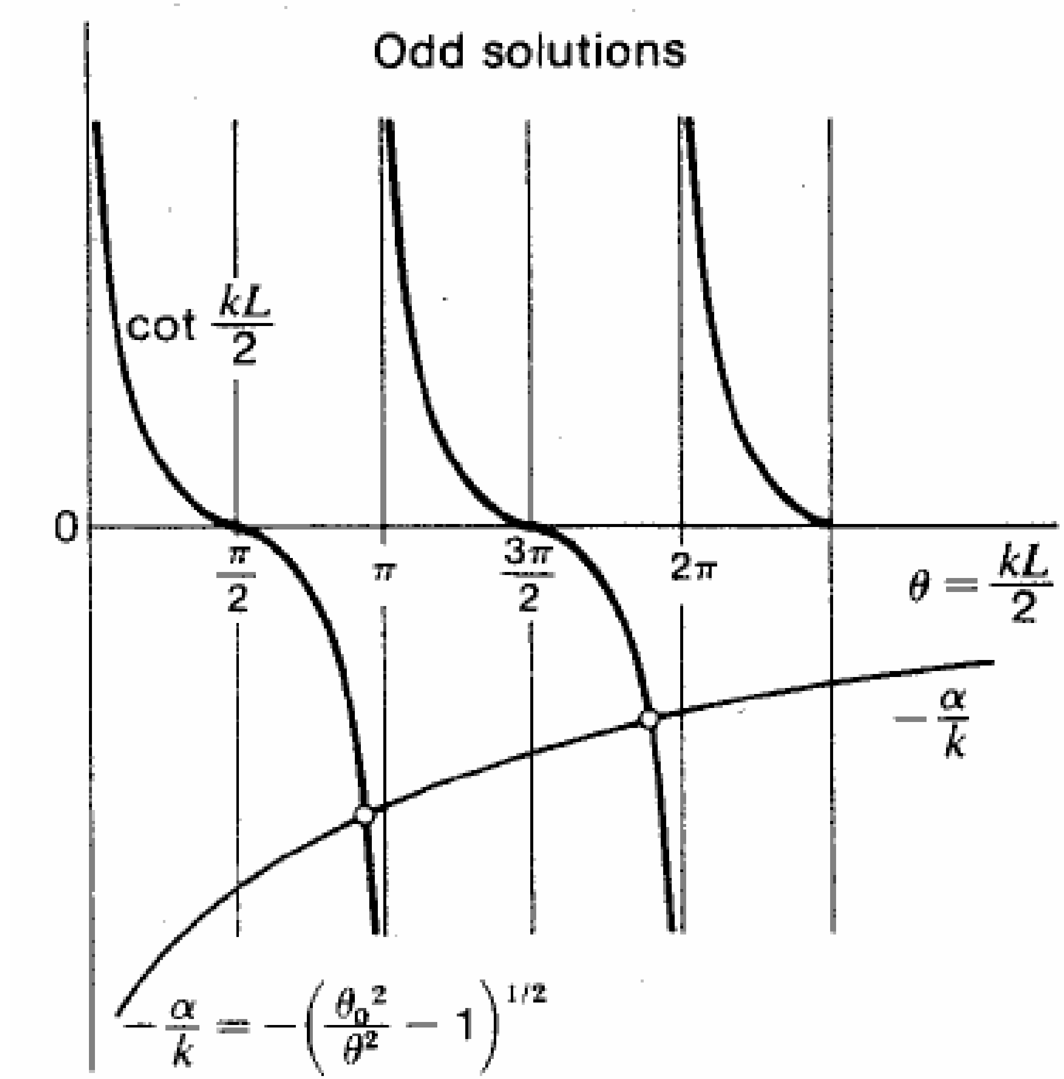

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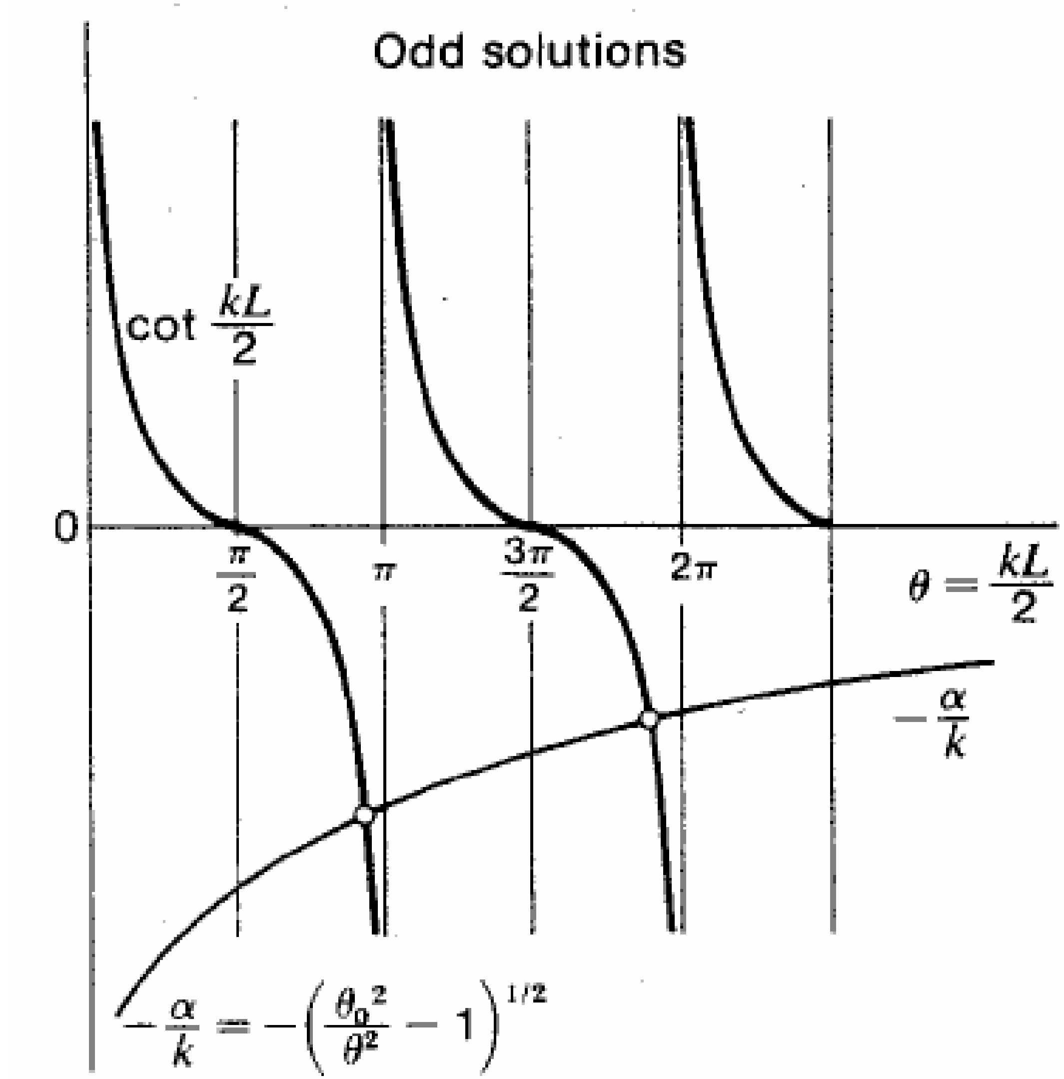

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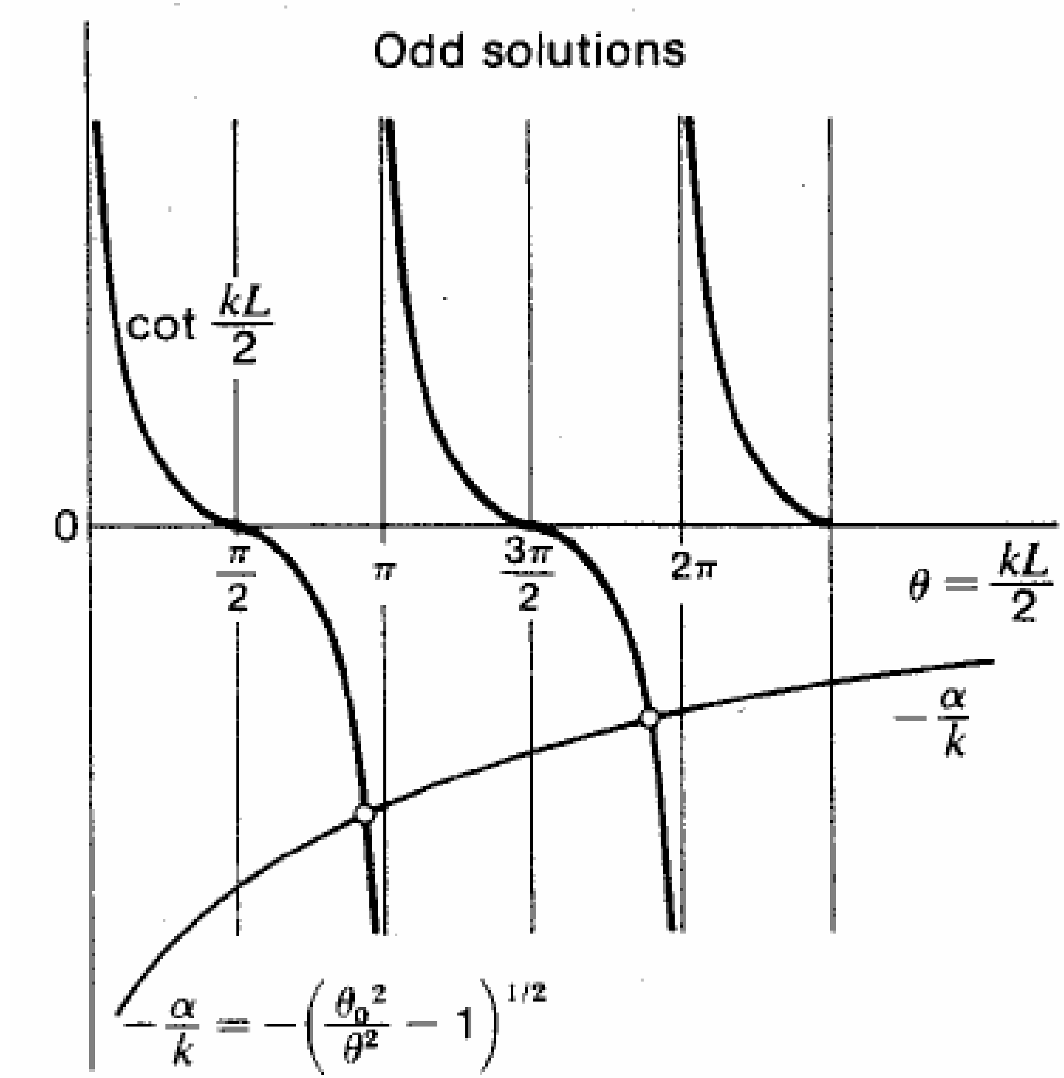

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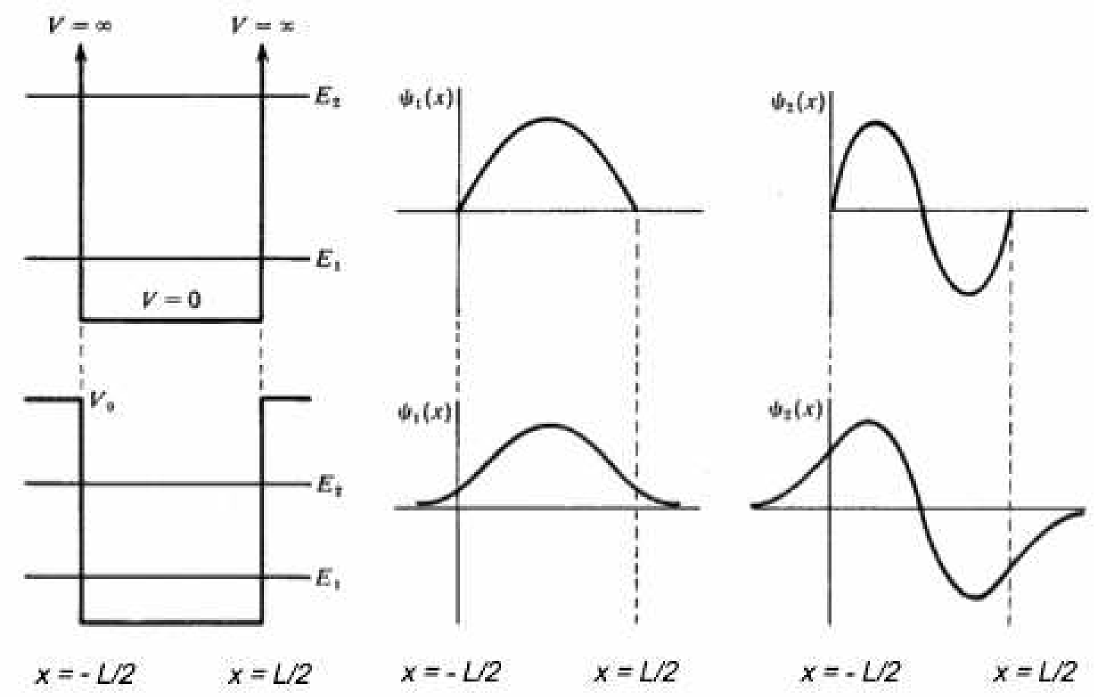
Compare the FPW and the IPW



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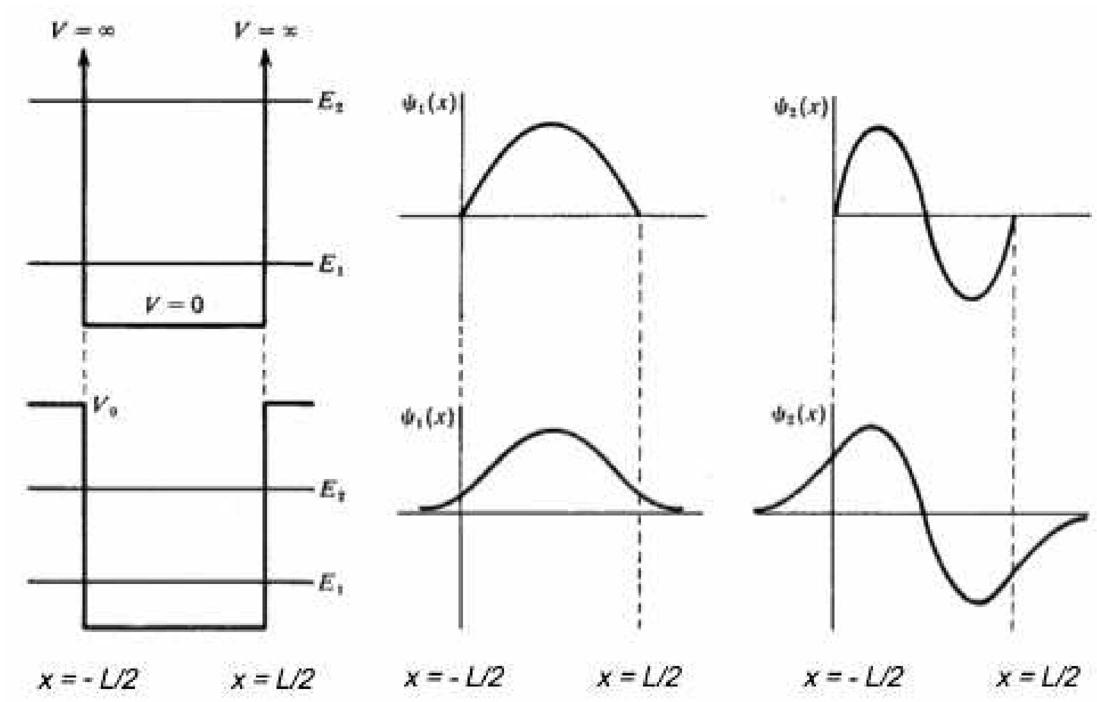

Infinite well:

$\psi(x)$ confined to the well

$k_n = \frac{n\pi}{L}$

infinite tower of states

no unbound states

Finite well:

$\psi(x)$ spreads out beyond the well

k_n and energies lower

finite tower of states

unbound states when $E > V_0$

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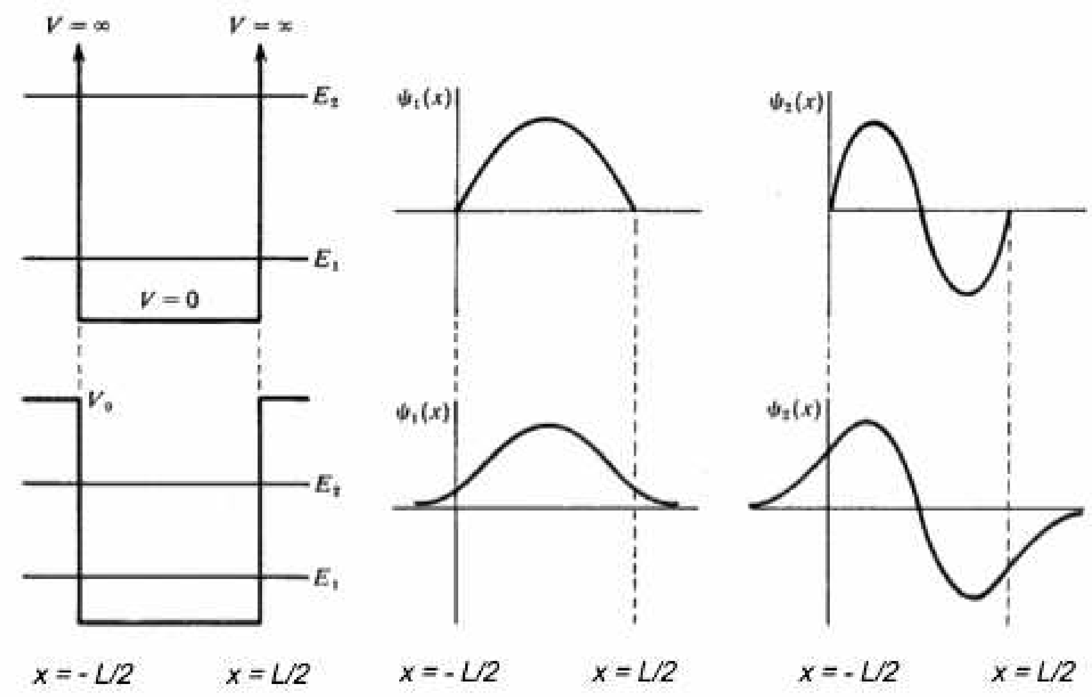

Infinite well:	Finite well:
$\psi(x)$ confined to the well	$\psi(x)$ spreads out beyond the well
$k_n = \frac{n\pi}{L}$	k_n and energies lower
infinite tower of states	finite tower of states
no unbound states	unbound states when $E > V_0$

The energy levels in the FPW are lower because the wavefunction spreads out (by penetrating the classically forbidden region) and therefore reduces its KE.

Quantum Tunnelling

Quantum Tunnelling

At $x > L/2$ the wavefn $\psi(x) \propto e^{-\alpha x}$;
at $x < -L/2$ the wavefn $\psi(x) \propto e^{\alpha x}$.

$$\alpha^2 = \frac{2m(V_0 - E)}{\hbar^2} > 0$$

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  $\Rightarrow$ Requiring a "reasonable behaviour" of the wavefunction leads to a (classically) "crazy" phenomenon of tunnelling

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- quantum states in symmetric potentials (w.r.t. reflections $x \longrightarrow -x$) are either symmetric (i.e., even parity), with an even number of nodes, or else antisymmetric (i.e, odd parity), with an odd number of nodes

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- when $V = V(x)$, both bound and continuous states are stationary, i.e, the time-dependent wavefunctions are of the form $\Psi(x,t) = \psi(x) \exp\left(-\frac{i}{\hbar}Et\right)$